# Counting Words:
## LNRE Modelling

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# Outline

# Where we are at

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- We discussed random samples and expected values
- We showed how to estimate model parameters by comparing observed / expected frequency spectrum
- ➥ We need an efficient way to calculate expected values
  - for random samples of arbitrary size $N$
  - given a model of the population type probabilities $\pi_k$

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- Mathematically, $\mathrm{E}[V_m(N)]$ is the limit of this expression for $n \to \infty$ (but you can just think of $n$ as very large)

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- Taking the sum over all types $w_k$ and dividing by $n$:

$$\mathrm{E}\big[V_m(N)\big] = \sum_k p_{k,m} = \sum_k \binom{N}{m} (\pi_k)^m (1 - \pi_k)^{N-m}$$

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  - arguably, statistically most appropriate
- But mathematically simpler to use **Poisson expectation**:

$$\mathrm{E}\big[V_m(N)\big] = \sum_k \frac{(N\pi_k)^m}{m!} e^{-N\pi_k}$$

  - here, we sum over samples of various sizes close to $N$

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- **Practical:**
  - When $N$ is large and $\pi$ small (as with word frequency distributions), Poisson probabilities are a very good approximation to binomial probabilities
- In lexical statistics, word frequency distribution models almost always use Poisson expectations

# Poisson expectations for $V_m$ and $V$

$$\mathrm{E}\big[V_m(N)\big] = \sum_k \frac{(N\pi_k)^m}{m!} \cdot e^{-N\pi_k}$$

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- $\mathrm{E}[V]$ sums over probabilities that $w_k$ occurs at least once
- ☞ Now we need to plug in population model for $\pi_k$ (we will use the Zipf-Mandelbrot model, of course)

# Plugging in the population model

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... and to a computer

# Outline





Computing expectations from the population model

The type density function and LNRE modeling

Zipf-Mandelbrot as LNRE model

Wrapping up

# The bad news

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- Are we stuck?

# An idea. . .

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- More robust view on the data by pooling types with the same frequency ➜ frequency spectrum
- Perhaps we can use a similar approach for the probabilities of the population model?

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  - e.g. for $\pi_k = .02501$ vs. $\pi_l = .02504$
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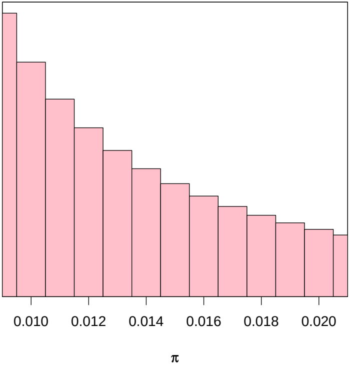


- $L = 1000$ cells
- cell $j$ represents types with $\pi_k \approx j/L$
- cell count $c_j$ = *area* of bar in histogram

# Plugging in, 2nd attempt

- Produce histogram with $L$ cells (e.g., $L = 1000$)
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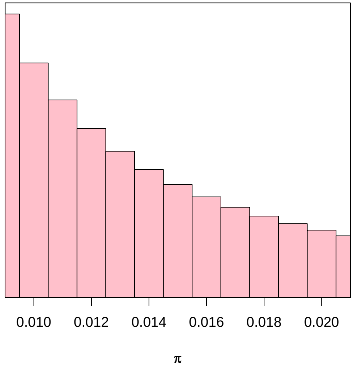
# Refining the histogram





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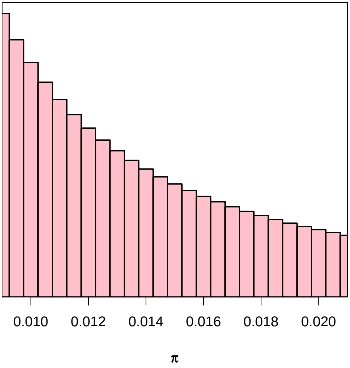


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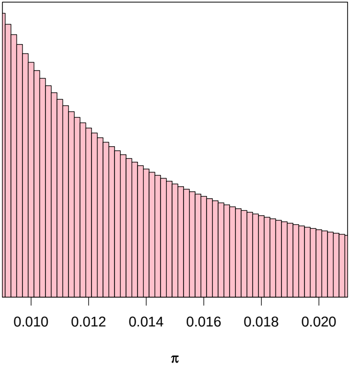


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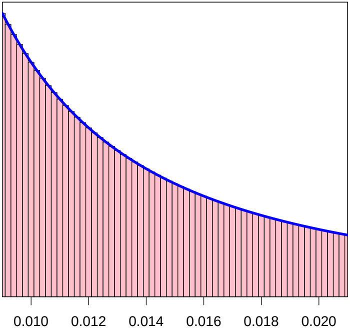


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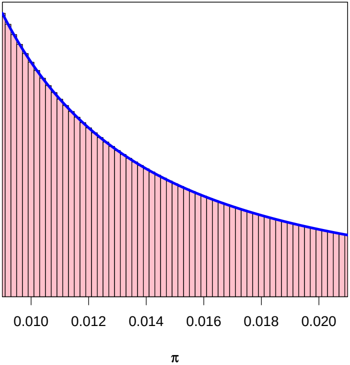


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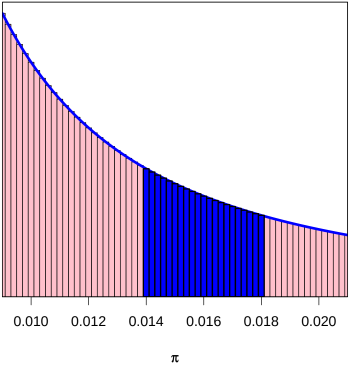


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$$= \int_A^B g(\pi)\, d\pi$$

# The integral form of expectations





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- Beautiful! :-)

# Summary time

## What did we just do?

- Initial formula was too complex
- Histogram approximation: simpler but coarse
- Get nuances back by increasing number of cells
- . . . but this time we end up with a convenient integral that we can compute efficiently!

# LNRE models

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- We can plug in any function $g$ defined on $[0, 1]$
- Population model expressed in terms of a **type density function** $g$ is what we call a **LNRE model** (for Large Number of Rare Events, Baayen 2001)

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- Moreover, we want to use a function that can be derived from a plausible population model, e.g. Zipf-Mandelbrot

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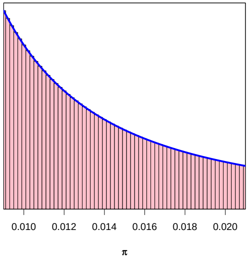
➥ Goal is to find a function $g(\pi)$ that corresponds to a very fine histogram of the ZM (or fZM) type population



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  - we don't even know which family of functions to use
  - there must be a better way!
- Luckily, there is an analytical solution

# Summary of the next few steps . . .

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- Out comes ZM formulated in terms of $g(\pi)$
- And now . . . another detour (sorry!)

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- From this equation we will be able to work out $G$
- With the help of $G$ we can then derive the LNRE formulation of ZM in terms of a type density function $g$
  - NB: upper case $G$ stands for the type distribution, lower case $g$ for the type density function (standard notation)

# Sneak preview: from $G$ to $g$

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  - $G(\rho)$ = number of types with $\rho \leq \pi_k$
  - there are no types with $\pi_k > 1$

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- This is the second fundamental theorem of calculus
- Intuitively:
  - If you increase $\rho$, say from $\rho$ to $\rho + x$, $G$ *decreases* (fewer types ➜ minus sign)
  - The *amount* by which it decreases (number of types between $\rho$ and $\rho + x$) is proportional to $g(\rho)$

# Calculating $G$ from the Zipf-Mandelbrot law

- According to the ZM law, for $\rho = \pi_k$ there are exactly $k$ types with $\pi \geq \rho$ (viz. the types $w_1, \ldots, w_k$), i.e.:

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☞ Find a function $G$ that satisfies this equation
  - err ...

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- We can now define $G$ by

$$G(\rho) := C^{\frac{1}{a}} \cdot \rho^{-\frac{1}{a}} - b$$

and have found a function that satisfies $G(\pi_k) = k$

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- $\alpha = \frac{1}{a}$ replaces ZM's $a$ as "slope" parameter ($0 < \alpha < 1$)
- $C^*$ is normalizing constant determined from constraint

$$\int_0^1 \pi \cdot g(\pi)\, d\pi = 1$$

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- Obvious choice: $\pi_1$, but for mathematical reasons the threshold parameter $B$ *close* rather than equal to $\pi_1$
- Surprise, surprise: $B = \dfrac{a-1}{b}$
  - ☞ $b$ is back!

# The LNRE ZM model

$$g(\pi) = \begin{cases} C \cdot \pi^{-\alpha-1} & 0 \leq \pi \leq B \\ 0 & \pi > B \end{cases}$$

- shape parameter $0 < \alpha < 1$ ("slope")
- (upper) cutoff parameter $0 < B \leq 1$
- $C = \dfrac{1-\alpha}{B^{1-\alpha}}$
- relation to Zipf-Mandelbrot law:

$$a = \frac{1}{\alpha} \qquad S = \infty$$

$$b = \frac{1-\alpha}{B \cdot \alpha}$$

# Expectations under the LNRE ZM model





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\begin{aligned}
\mathrm{E}\big[V_m(N)\big] &= \int_0^1 \frac{(N\pi)^m}{m!} e^{-N\pi} g(\pi)\, d\pi \\
&= \frac{C}{m!} \cdot \int_0^B (N\pi)^m e^{-N\pi} \pi^{-\alpha-1}\, d\pi \\
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- The (lower) incomplete **Gamma function** $\gamma$ is a so-called **special function** ➜ well-understood by mathematicians
- $\gamma$ and $m! = \Gamma(m+1)$ can be computed efficiently
- This and several similar properties make the LNRE formulations of ZM and fZM convient and robust

# The LNRE fZM model





$$g(\pi) = \begin{cases} C \cdot \pi^{-\alpha-1} & A \leq \pi \leq B \\ 0 & \text{otherwise} \end{cases}$$

- shape parameter $0 < \alpha < 1$ ("slope")
- cutoff parameters $0 < A < B \leq 1$
  - fZM with $A = 0$ ➜ ZM model
- $C = \dfrac{1-\alpha}{B^{1-\alpha} - A^{1-\alpha}}$
- relation to Zipf-Mandelbrot law:

$$a = \frac{1}{\alpha} \qquad S = \frac{1-\alpha}{\alpha} \cdot \frac{A^{-\alpha} - B^{-\alpha}}{B^{1-\alpha} - A^{1-\alpha}}$$

$$b = \frac{C}{B^{\alpha} \cdot \alpha}$$

# Outline

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- The components of a LNRE model:
  - Population model, expressed as family of **type density functions** (determines overall shape of distribution)
  - **Parameters** of the type density function (determine how steep the curve is and other aspects of its shape)
  - Formulas to compute **expectations** for $V$ and spectrum elements $V_m$ in samples of arbitrary size $N$ (we used Poisson sampling, but there are other options)

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- In order to *apply* LNRE model to real-life data you need a way to **estimate model parameters** (typically by matching expected and observed frequency spectrum)
- Aspects you might actively intervene in:
  - choose a LNRE model
  - details of parameter estimation (cost function etc.)

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in zipfR

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- `plot(spc.exp)`
  - ☞ plot expected spectrum